ON KUMARASWAMY INVERTED WEIBULL DISTRIBUTION AND ITS APPLICATION

AladesuyiAlademomi, Ajayi Bamidele
Department of Mathematics and Statistics, Federal Polytechnic Ado-Ekiti, Nigeria
Aladesuyialademomi@gmail.com, dele2304@yahoo.com

ABSTRACT: In this paper, a new three-parameter generalized version of the inverted Weibull distribution called Kumaraswamy inverted Weibull (KIW) distribution. The new distribution is quite flexible and can have a decreasing, increasing, and bathtub-shaped failure rate function depending on its parameters making it effective in modeling survival data and reliability problems. The maximum likelihood function of the new distribution was derived. Some comprehensive properties of the new distribution, such as closed-form expressions for the density, cumulative distribution, hazard rate function, the *i*th order statistics were provided. At the end, in order to show the capability of KIWdistribution over its sub models, an application to a real dataset illustrates its potentiality.

***Key words*:** kumaraswamyinverted Weibull distribution, Maximum likelihood estimation, Bathtub-shaped failure rate.

1. Introduction

Modelling of interrelationship among naturally occurring phenomena is made possible by the use of distribution function and their properties. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies.The paper by Kumaraswamy proposed a new probability distribution for double bounded random processes with hydrological applications. This new family of distribution most especially its probability density function has been found to have the same properties as the beta distribution but has some advantages in terms of tractability.

2. Kumaraswamy distribution

The kumaraswamy distribution on the interval (0,1) has the probability density function, $f(x)$ corresponding to (1) and cumulative distribution function, $F(x)$ which takes the form (2) with two shape parameter , a> 0 $and\ b$ >0 defined by,

$$f(x) = abx^{a-1}(1-x^a)^{b-1} \tag{1}$$

And

$$F(x) = 1-(1-x^a)^b \tag{2}$$

Combining the work of Eugene et al (2002) and Jones (2004) to construct a new class of kumaraswamy generalized (KwG) distribution can be obtained. From an arbitrary parent cumulative density function, F(x), the cumulative density function, $G(x)$ of the Kumaraswamy Generalized distribution is defined by

$$G(x) = 1-(1-F(x)^a)^b \tag{3}$$

Where a>0 and $b>0$ are two additional parameters whose role is to introduce Skewness and vary the tail weights. Because of its tractability, the kumaraswamy distribution function (Kw) distribution can be used quite effectively even if the data were censored.

Correspondingly, the density function of this family has a very simple form given by

$$g(x) = abf(x)F(x)^{a-1}(1-F(x)^{a})^{b-1} \tag{4}$$

Several generalized distributions from (4) have been defined and investigated in the literature including the Kumaraswamy Weibull distribution by Cordeiro *et al*. (2010), the Kumaraswamy generalized gamma distribution by de Castro *et al*. (2011) and the Kumaraswamy generalized half-normal distribution by Cordeiro *et al*. (2012).

3. Kumaraswamy Inverted Weibull Distribution (KIWD)

We say that the random variable X has a standard inverted Weibull distribution (IWD) if its distribution function takes the following form:

$$G(x) = e^{-x^{-\beta}} \qquad x>0, \tag{5}$$

The pdf of Inverted Weibull distribution is given by

$$g(x) = \beta x^{-\beta} e^{-x^{-\beta}} \tag{6}$$

Now using (5) and (3) we have the cdf of a (KIWD) given by

$$G(x) = 1-\left\{1-\left(e^{-x^{-\beta}}\right)^{a}\right\}^{b} \quad x\geq 0, a, b, \beta>0 \tag{7}$$

From Taylor series expansion given as:

$$(1-m)^{z} = \sum_{i=1}^{\infty}(-1)^{i}\binom{z}{i}m^{i} \tag{8}$$

Then equation (7) we transform to

$$G(x) = 1-\sum_{i=1}^{n}(-1)^{i}\binom{b}{i}\left(e^{-x^{-\beta}}\right)^{a} \tag{9}$$

Also putting (5) and (6) in (4) we have the cdf of KIW distribution

$$g(x) = ab\beta x^{-\beta}\left\{e^{-x^{-\beta}}\right\}^{a}\left[1-\left\{e^{-x^{-\beta}}\right\}^{a}\right]^{b-1} \tag{10}$$

Using equation (8), equation (9) can be simplified as:

$$g(x) = ab\beta x^{-\beta}\sum_{i=1}^{\infty}(-1)^{i}\binom{b-1}{i}\left\{e^{-x^{-\beta}}\right\}^{a(i+1)} \tag{11}$$

Figure (1) and (2) depict the behaviour of the distribution for some parameters values, $a=1.6, b=2.5, b_1=1.8$.

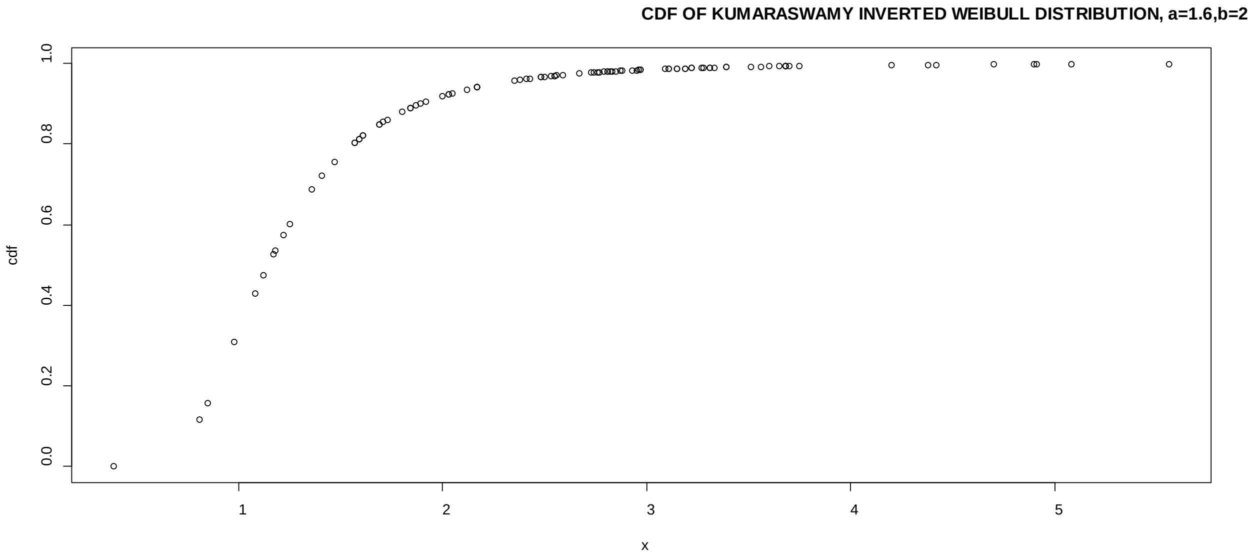

Figure 1. The graph of the cdf of kumaraswamy Inverted Weibull distribution

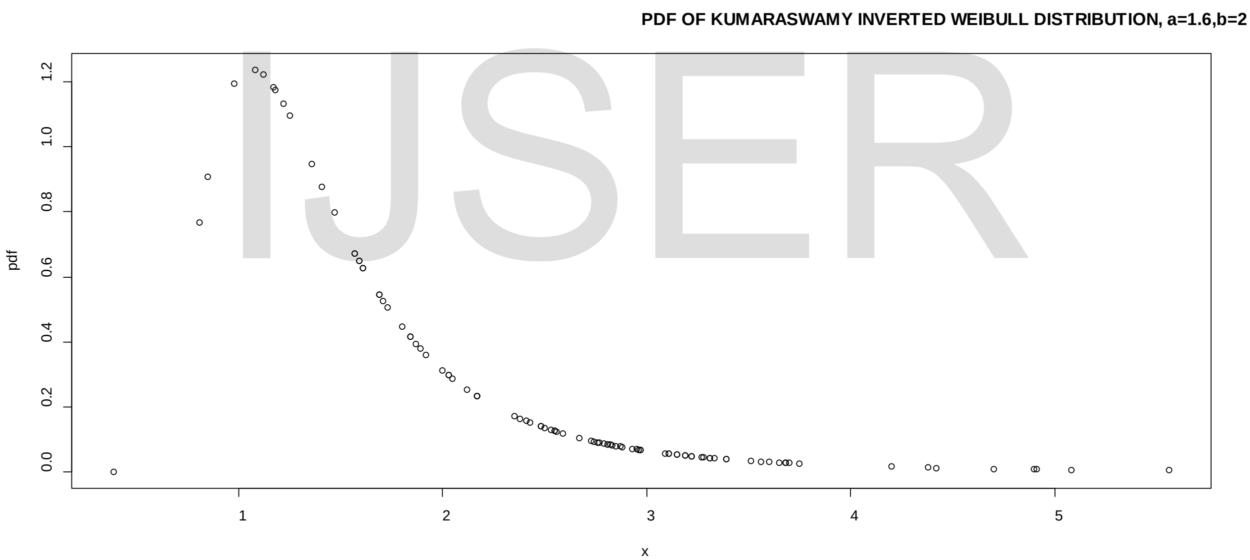

Figure2. The graph of the pdf of kumaraswamy Inverted Weibull distribution

The plot of probability density function and the Cumulative density function shows in the diagram above clearly indicates that the Kumaraswamy Inverted Weibull distribution is more flexible than the inverted Weibull distribution

4. Statistical Properties:

Asymptotic Behavior

We seek to investigate the behaviour of the model in equation (10)

$$\lim_{x\mapsto 0} ab\beta x^{-\beta}\left\{e^{-x^{-\beta}}\right\}^{a}\left[1-\left\{e^{-x^{-\beta}}\right\}^{a}\right]^{b-1}$$
$$=0$$

5. Hazard Rate function

The hazard rate function can be obtained by using,

$$h(x)=\frac{g(x)}{1-G(x)}$$

$$h(x)=\frac{ab\beta x^{-\beta}\left\{e^{-x^{-\beta}}\right\}^{a}\left[1-\left\{e^{-x^{-\beta}}\right\}^{a}\right]^{b-1}}{\left\{1-\left(e^{-x^{-\beta}}\right)^{a}\right\}^{b}} \tag{12}$$

This can be simplified to obtain,

$$h(x)=\frac{ab\beta x^{-\beta}\left\{e^{-x^{-\beta}}\right\}^{a}}{1-\left(e^{-x^{-\beta}}\right)^{a}} \tag{13}$$

The equation (12) is the hazard of the kumaraswamy Inverted Weibull distribution and also called the Kumaraswamy Inverted Weibull Model.

When the value of $a=b=1$, we have,

$$h(x)=\frac{\beta x^{-\beta}e^{-x^{-\beta}}}{1-e^{-x^{-\beta}}} \tag{14}$$

The above is the hazard of Inverted Weibulldistribution or Inverted Weibull model.

6. Reliability function

Thereliability of a function is defined by:

$$R(x)=1-G(x) \tag{15}$$

For Kumaraswamy inverted Weibull, the Reliability function is given as,

$$R(x)=\left\{1-\left(e^{-x^{-\beta}}\right)^{a}\right\}^{b} \tag{16}$$

Therefore

$$R(x)=\sum_{i=1}^{n}(-1)^{i}\binom{b}{i}\left(e^{-x^{-\beta}}\right)^{a} \tag{17}$$

ISSN 2229-5518

7. The Quartile:

The quartile x_u of order u for the KIW distribution is given by the solution of

$$x_u = \left[-\left\{ \ln(1 - \{1-u\}^{\frac{1}{b}} \right\}^{\frac{1}{a}} \right]^{-\frac{1}{\beta}} \tag{18}$$

Proof

Let $u = F(x) = 1 - \left\{ 1 - \left(e^{-x^{-\beta}} \right)^a \right\}^b$

Then we have

$$1 - u = \left\{ 1 - \left(e^{-x^{-\beta}} \right)^a \right\}^b$$

This result in

$$1 - (1-u)^{\frac{1}{b}} = \left(e^{-x^{-\beta}} \right)^a \tag{19}$$

Finally,

$$x_u = \left[-\left\{ \ln(1 - \{1 - u\}^{\frac{1}{b}} \right\}^{\frac{1}{a}} \right]^{-\frac{1}{\beta}} \tag{20}$$

8.0 Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. The density $f_{i:n}(x)$ of the ith order statistics for $= 1, \dots, n$, from $i.i.d$ random variables $X_1, X_2, \dots X_n$ that follows any kumaraswamy generalized distribution is given by,

$$g_{i:n}(x) = \frac{\beta x^{-\beta} e^{-x^{-\beta}}}{B(i:n-i+1)} \left(e^{-x^{-\beta}} \right)^{i-1} \left[1 - \left\{ 1 - \left(e^{-x^{-\beta}} \right)^a \right\}^b \right] \left\{ \left(e^{-x^{-\beta}} \right)^a \right\}^{b(n-i+1)-1} \tag{21}$$

9.0 Estimation of Statistical Inference

Let $x_1, x_2, \dots, x_n$ be random variable distributed according to (8) the likelihood function of a vector of parameters given as $\Omega(a, b, \beta)$.

$$l(\Omega) = n\{\log(a) + \log(b)\} + nlog(\beta) - a \sum_{i=1}^{n} x^{-\beta} + (b-1) \sum_{i=1}^{n} \left[1 - e^{-x^{-\beta}} \right]^a \tag{22}$$

Then the score vector $\nabla l = \frac{\delta l}{\delta a}, \frac{\delta l}{\delta b}, \frac{\delta l}{\delta \beta}$ has components,

$$\frac{\delta l}{\delta a} = \frac{n}{a} + \sum_{i=1}^{n} x^{-\beta} + \sum_{i=1}^{n} \left[1 - e^{-x^{-\beta}} \right]^a log\left(1 - e^{-x^{-\beta}} \right) \tag{23}$$

$$\frac{\delta l}{\delta b} = \frac{n}{b} + \sum_{i=1}^{n} \left[1 - e^{-x^{-\beta}}\right]^{a} \tag{24}$$

$$\frac{\delta l}{\delta \beta} = \sum_{i=1}^{n} \frac{n}{\beta} - a \sum_{i=1}^{n} x e^{-x^{-\beta}} \left[1 - e^{-x^{-\beta}}\right]^{a} \tag{23}$$

10. Application

To illustrate the new results presented in this paper, we fit the KIW distribution to an uncensored data set from Nichols and Padgett,(2006) considering 100 observations on breaking stress of carbon fibres (in Gba). The data are as follows : 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59,3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 2.82, 2.05, 3.65.These data were previously studied by Souza *et al*. (2011) for beta Frechet (BF), exponentiated Frechet (EF) and Frechet distributions. In the following, we shall compare the proposed KGM and its sub-model (GM) with several other three- and four-parameter lifetime distributions, namely: the Zografos-Balakrishnan log-logistic (ZBLL) (Zografos and Balakrishnan, 2009), the beta Frechet (BF) (Nadarajah and Gupta, 2004 and Souza *et al*., 2011) and recently the Kumaraswamy Pareto (KP) (Bourguignon *et al*., 2013) models with corresponding densities:

$$ZBLL: f_{ZBLL}(x, a, \beta, \theta) = \frac{\beta}{\theta^{\beta}\tau(a)} x^{\beta-1}(1 + (\frac{x}{\theta})^{\beta})^{-2} \left[\ln(1 + (\frac{x}{\theta})^{\beta})\right]^{a-1} \qquad x > 0$$

$$BF: f_{BF}(x, a, b, \theta, \beta) = \frac{\beta\theta^{\beta}}{B(a,b)} x^{-(\beta+1)} e^{-a(\frac{\theta}{x})^{\beta}} (1 - e^{-(\frac{\theta}{x})^{\beta}})^{b-1} \qquad x > 0$$

$$KP: f_{KP}(x, a, b, \theta, \beta) = ab\beta\theta^{\beta} x^{-(\beta+1)} \left[1 - (\frac{\theta}{x})^{\beta}\right]^{a-1} \left[1 - (1 - (\frac{\theta}{x})^{\beta})^{a}\right]^{b-1} \qquad x > 0$$

Where $a, b, \beta, \theta > 0$

Table1 gives the descriptive statistics of the data and Table 2 lists the MLEs of the model parameters for KIW, GM, BF, KP, ZBLL, BF and EF distributions, the corresponding errors(given in parenthesis) and the statistics $l(\hat{\theta})$ (where $l(\hat{\theta})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates), Akaike information criterion (AIC), the Bayesian information criterion (BIC), Consistent Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC). Since the KIW distribution has the lowest $l(\hat{\theta})$, AIC, BIC, CAIC and HQIC values among all other models and so it could be chosen as the best model.

ISSN 2229-5518

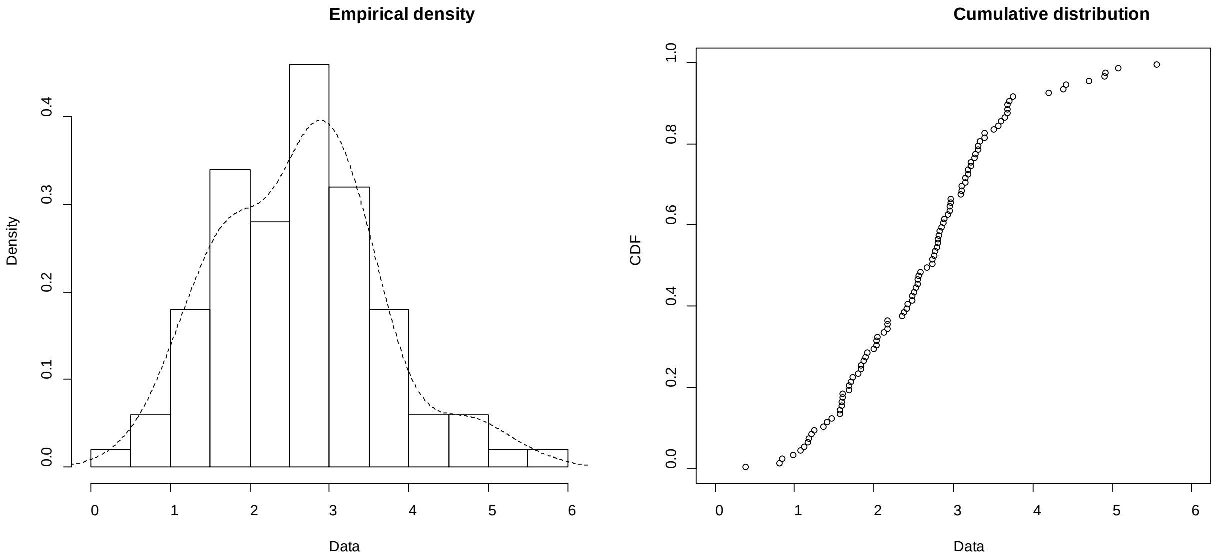

Fig. 4. The graph of the Empirical density and the cumulative distribution function of the carbon data

Table 1 Descriptive Statistics on Breaking stress of Carbon fibres.

Min	Q_1	Median	$mean$	Q_3	Max	$kurtosis$	Skewness
0.390	1.840	2.700	2.640	3.220	5.560	0.17287	0.37378

Table2 MLEs(standard error in parenthesis) and the statistics $l(\hat{\theta})$, AIC, BIC and $HQIC$

$Model$	$Estimates$					$l(\hat{\theta})$	AIC	BIC	HQIC
KIW $(a,b,\lambda,\alpha,\beta)$	13.005(5.5 96)	0.985(0.25 5)	13.082(7. 114)			-8.678	23.357	29.786	25.886
KP (a,b,θ,β)	4.69523 (0.502)	236.2335 (149.552)	0.39 -	0.19204 (0.045)	-	-166.751	339.502	347.318	338.084
$ZBLL$ (a,θ,β)	1.55009 (0.104)	1.90903 (0.0093)	3.61259 0.288	-	-	-162.913	331.826	339.642	330.408
BF (a,b,θ,β)	0.42934 (0.236)	138.0664 (113.552)	34.38484 (21.52)	0.72474 (0.19)	-	-142.866	293.733	304.154	291.842
EF (b,θ,β)	52.0491 (31.954)	26.1730 (14.666)	0.6181 (0.0897)	-	-	-145.087	296.174	303.989	294.755
IW (β)	2.806 (2.723)	1.76902 (0.114)	-	-	-	-58.482	118.963	121.107	119.029

REFRENCES

[1] A. Flair, H. Elsalloukh, E. Mendi and M. Milanova. The exponentiated inverted Weibull Distribution, Appl. Math. Inf. Sci. **6**, No. 2, 167-171 (2012)

[2] G.S. Mudholka, D.K. Srivastava and M. Freimer, The exponentiated Weibull family: a real analysis of the bus motor failure data.

[3] G. S. Mudholkar and A. D. Hutson, Exponentiated Weibull family: some properties and flood data application, Commun. Statistical Theory and Method.**25**, 3050-3083 (1996).

[4] G.R. Aryal, and Ch. P. Tsokos, (2011) Transmuted Weibull Distribution: A Generalization of the Weibull probability distribution, European journal of pure and applied mathematics Vol. 4, No.2, 89- 102.

[5] Cordeiro, G.M., E.M.M.Ortega and S. Nadarajah, 2010. The Kumaraswamy Weibull distribution with application to failure data. J. Frankl. Inst., 347: 1399-1429.

[6] Chukwu A.U. & Ogunde A. A. (2015), 'On the Beta Makeham Distribution and its Applications', American Journal of Mathematics and Statistics 2015, 5(3): 137-143

[7] El-Gohary, A. & Al-Otaibi, A. N. (2013), 'The generalized Gompertz distribution', Applied Mathematica Modeling 37(1-2), 13–24.

[8] Nadarajah, S. and S. Kotz, 2003. The exponentiated Fréchet distribution.Available at Interstat. statjournals net, 0312001.

[9] Nichols, M.D. and W. J. Padgett, 2006. A bootstrap control chart for Weibull Percentiles. Quality and Reliability Engineering International, 22: 141-151.

[10] Shahbaz, M.Q., S. Shahbaz and N.S. Butt, 2012. The Kumaraswamy inverse Weibull distribution. Pakistan Journal of Statistics and Operation Research, 8: 479-489.